

SOME MONOTONICITY RESULTS FOR GENERAL SYSTEMS OF NONLINEAR ELLIPTIC PDES

JULIEN BRASSEUR AND SERENA DIPIERRO

ABSTRACT. In this paper we show that minima and stable solutions of a general energy functional of the form

$$\int_{\Omega} F(\nabla u, \nabla v, u, v, x) dx$$

enjoy some monotonicity properties, under an assumption on the growth at infinity of the energy.

Our results are quite general, and comprise some rigidity results which are known in the literature.

CONTENTS

1. Introduction	1
1.1. Monotonicity and 1-dimensional symmetry for systems	1
1.2. The mathematical setting	3
1.3. A monotonicity result for minimizers of the energy functional	5
1.4. A monotonicity result for stable solutions	5
1.5. Organization of the paper	6
2. Stability property	6
3. Local analysis	7
4. Perturbations at infinity	13
5. Proof of Theorem 1.5	15
6. Proof of Theorem 1.10	16
7. Some applications	17
7.1. Two-stated mixture of Bose-Einstein condensate	17
7.2. General systems with p -Laplacian type operators	18
7.3. Systems involving the fractional Laplacian	20
Acknowledgments	21
References	21

1. INTRODUCTION

1.1. Monotonicity and 1-dimensional symmetry for systems. In this paper we study monotonicity properties of minima and stable solutions of a quite general

2010 *Mathematics Subject Classification.* 35J50.

Key words and phrases. Monotonicity results, stable solutions, domain perturbations.

energy functional of the type

$$(1.1) \quad \int_{\Omega} F(\nabla u, \nabla v, u, v, x) dx,$$

where $\Omega \subset \mathbb{R}^n$.

Recent years have seen numerous ongoing research activities in investigating symmetry properties of systems of PDEs. A typical example is

$$(1.2) \quad \begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ u, v > 0. \end{cases}$$

which arise in phase separation for Bose-Einstein condensates with multiple states. We refer to [1, 2] and references therein for a derivation of this phase separation model.

In particular, in [1], the authors prove existence, symmetry and non-degeneracy for solutions to (1.2) in 1-dimension. They show that entire solutions are reflectionally symmetric, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x - x_0) = v(x_0 - x)$ for any $x \in \mathbb{R}$. They also establish a result which may be seen as the analogue of a celebrated conjecture of De Giorgi for problem (1.2) in dimension 2; more precisely, they show that monotone solutions to (1.2) in dimension 2 have one-dimensional symmetry, under the growth condition

$$u(x) + v(x) \leq C(1 + |x|),$$

for some $C > 0$.

On the other hand, the linear growth is the lowest possible for positive solutions to (1.2). Namely, if there exists $\alpha \in (0, 1)$ such that

$$u(x) + v(x) \leq C(1 + |x|)^\alpha,$$

then $u = v \equiv 0$, see [17].

In dimension 2, in [12] improves the result in [1] showing that if (u, v) is a monotone solution to (1.2) and has at most algebraic growth at infinity then it must be one-dimensional.

The monotonicity condition can be weakened in order to get the one-dimensional symmetry for solutions to (1.2). Indeed, it has been proved in [2] stable solutions to (1.2) are one-dimensional in \mathbb{R}^2 .

Symmetry results in dimension 2 for systems with more general nonlinearities have been achieved in [15]. Quasilinear (possibly degenerate) elliptic systems in \mathbb{R}^2 have also been considered in [9].

As for dimensions higher than 2, in [14] the authors prove that if (u, v) has algebraic growth and

$$\lim_{x_n \rightarrow \pm\infty} (u(x', x_n) - v(x', x_n)) = \pm\infty \quad \text{uniformly in } x' \in \mathbb{R}^{n-1},$$

then (u, v) only depends on x_n . Also, in [24, 25] it has been proved that if (u, v) has linear growth then it is one-dimensional.

Recently the nonlocal counterpart of (1.2) has been also investigated, see e.g. [20, 21, 23], and some symmetry results have been obtained in [10] for a quite general system of nonlocal PDEs of the form

$$\begin{cases} (-\Delta)^{s_1} u = F_1(u, v), \\ (-\Delta)^{s_2} v = F_2(u, v), \end{cases}$$

where F_1 and F_2 denote the derivatives of a function $F \in C_{loc}^{1,1}(\mathbb{R}^2)$ with respect to the first and the second variable, respectively; $s_1, s_2 \in (0, 1)$ and, given $s \in (0, 1)$, $(-\Delta)^s$ stands for the fractional Laplace operator

$$(-\Delta)^s u(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where p.v. denotes the Cauchy principal value (see [8] for the definition and further details). See also [16] for symmetry results for nonlocal systems of equations.

Aim of this paper is to provide monotonicity results for minima and stable solutions of energy functionals of the form (1.1) (thus embracing all the systems considered above). We will consider the case in which both the domain Ω and the functional F are invariant under translations in the e_n -direction, and we will deal with symmetry properties of minima or stable solutions in the class of functions which are obtained by piecewise Lipschitz domain deformations in the e_n -direction (see Definitions 1.1 and 1.2). We will also require some mild assumptions on F and a growth condition (see (1.5)).

The key tool of our proofs rely on a technique introduced in [19] to study the regularity of fractional minimal surfaces in dimension 2 (see [5] where these object were introduced), and developed in [18] to work in a more general setting. See also [6, 11] where these techniques have been used in the context of free boundary problems.

Here we adapt the new strategy of [18] to the case of general systems of equations. The idea is to look at the stability inequality without dealing with its precise form (which, in some cases, can be very hard to handle), and simply compare the energies of the couple (u, v) to that of its translations. For this, one also needs to modify the “translated” couple at infinity to make it a compact perturbation of (u, v) . Here the growth condition comes into play and ensures that the energy of the perturbed couple can be made arbitrarily close to the energy of (u, v) . Moreover, if u and v are not monotone in the e_n -direction, then one can modify locally the perturbed couple in order to get lower energy, but this is in contradiction with the minimality of (u, v) .

This strategy allows to deal with quite general form of energy functionals.

We now introduce the setting in which we will work and give precise definitions and statements of our results.

1.2. The mathematical setting. We consider a domain $\Omega \subset \mathbb{R}^n$ and study the symmetry properties of minima for functionals of the type

$$\mathcal{E}(u, v) := \int_{\Omega} F(\nabla u, \nabla v, u, v, x) dx.$$

We assume that the domain Ω and the functional F are invariant under translations in the e_n -direction, namely

$$\Omega = V \times \mathbb{R}, \quad V \subseteq \mathbb{R}^{n-1},$$

and F does not depend on the x_n -coordinate.

We will denote a point in Ω by $x = (x', x_n) \in V \times \mathbb{R}$.

We also suppose that F is convex with respect to the first two variables. Precisely, we assume that

$$(1.3) \quad F = F(p_1, p_2, z_1, z_2, x') \in C(\mathbb{R}^{2n} \times \mathbb{R}^2 \times V),$$

and F is C^2 and uniformly convex in p_1 and p_2 at all p_1, p_2 with $p_1 \cdot e_n \neq 0$ and $p_2 \cdot e_n \neq 0$, for all $(z_1, z_2, x') \in \mathbb{R}^2 \times V$. Finally, we assume that there exists a constant $C > 0$ such that

$$(1.4) \quad \begin{aligned} |F_{p_1 p_1}(p_1 + q_1, p_2, z_1, z_2, x')| &\leq C |F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')|, \\ |F_{p_2 p_2}(p_1, p_2 + q_2, z_1, z_2, x')| &\leq C |F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')|, \\ |F_{p_1 p_2}(p_1 + q_1, p_2 + q_2, z_1, z_2, x')| &\leq C |F_{p_1 p_2}(p_1, p_2, z_1, z_2, x')|, \end{aligned}$$

for any $p_1, p_2, q_1, q_2 \in \mathbb{R}^n$ with $|q_1| \leq |p_1 \cdot e_n|/4$ and $|q_2| \leq |p_2 \cdot e_n|/4$. We will make these assumptions throughout the paper.

Given $R > 0$, we consider the following energy functional

$$\mathcal{E}_R(u, v) := \int_{\Omega \cap B_R} F(\nabla u(x), \nabla v(x), u(x), v(x), x') dx,$$

where, as usual, B_R denotes the ball of radius R centered at the origin.

Following the line in [18], we want to study the symmetry properties of minimal or stable solutions for the functional above among suitable perturbations, obtained by domain deformations in the direction given by e_n . To do this, we recall the following definitions introduced in [18].

Definition 1.1. Given a function w , an e_n -Lipschitz deformation of w in B_R is a function \bar{w} defined by

$$\bar{w}(x) = w(x + \varphi(x)e_n) \quad \text{for any } x \in \Omega,$$

where φ is a Lipschitz function with compact support in B_R , with $\|\partial_n \varphi\|_{L^\infty(\mathbb{R}^n)} < 1$.

Definition 1.2. Given a function $w \in C^{0,1}(\Omega)$, a piecewise e_n -Lipschitz deformation of w in B_R is a function $\bar{w} \in C^{0,1}(\Omega)$ defined by

$$\bar{w}(x) = \bar{w}^{(i)}(x) \quad \text{for some } i \text{ (depending on } x \in \Omega),$$

where $\bar{w}^{(1)}, \dots, \bar{w}^{(m)}$ constitute a finite number of e_n -Lipschitz deformations of w in B_R . In this case, we write $\bar{w} \in D_R(w)$.

Also, if all $\bar{w}^{(i)}$ satisfy

$$\bar{w}^{(i)}(x) = w(x + \varphi^{(i)}(x)e_n) \quad \text{with } \|\varphi^{(i)}\|_{C^{0,1}(\Omega)} \leq \delta$$

for some δ , we write $\bar{w} \in D_R^\delta(w)$.

We recall also some elementary properties of a piecewise e_n -Lipschitz deformation, which follow easily from the definitions above:

Proposition 1.3. *The following properties hold true:*

- i) if $\bar{w}^1, \bar{w}^2 \in D_R^\delta(w)$, then $\min\{\bar{w}^1, \bar{w}^2\}, \max\{\bar{w}^1, \bar{w}^2\} \in D_R^\delta(w)$;
- ii) if $\bar{w} \in D_R^\delta(w)$, $\tilde{w} \in D_R^\delta(\bar{w})$, then $\tilde{w} \in D_R^{3\delta}(w)$;
- iii) if $\bar{w} \in D_R^\delta(w)$, then $\|\bar{w} - w\|_{L^\infty(\Omega)} \leq C\delta\|w\|_{C^{0,1}(\Omega)}$;
- iv) if $\bar{w} \in D_R^\delta(w)$, $w \in C^{1,1}(\Omega)$, then $\|\bar{w} - w\|_{C^{0,1}(\Omega)} \leq C\delta\|w\|_{C^{1,1}(\Omega)}$.

In the sequel we will also assume a growth condition on the functional \mathcal{E} . Namely, we suppose that there exists a constant $C > 0$ such that, for R sufficiently large,

$$(1.5) \quad \begin{aligned} \int_{\Omega \cap B_R} &|F_{p_1 p_1}(\nabla u, \nabla v, u, v, x')| |\nabla u|^2 + |F_{p_2 p_2}(\nabla u, \nabla v, u, v, x')| |\nabla v|^2 \\ &+ |F_{p_1 p_2}(\nabla u, \nabla v, u, v, x')| |\nabla u| |\nabla v| dx \leq CR^2. \end{aligned}$$

1.3. A monotonicity result for minimizers of the energy functional. The first result here deals with e_n -minimizers of \mathcal{E} . To state it, we give the following:

Definition 1.4. We say that (u, v) , with $u, v \in C^{0,1}(\Omega)$, is an e_n -minimizer for \mathcal{E} if for any $R > 0$ we have that $\mathcal{E}_R(u, v)$ is finite and

$$\mathcal{E}_R(u, v) \leq \mathcal{E}_R(\bar{u}, \bar{v}), \quad \text{for any } \bar{u} \in D_R(u) \text{ and any } \bar{v} \in D_R(v).$$

Now, we are in the position to state our first monotonicity result.

Theorem 1.5. Let $u, v \in C^1(\Omega)$, and let F satisfy (1.3) and (1.4). Suppose that (u, v) is an e_n -minimizer for the energy \mathcal{E} , and that the growth condition (1.5) is satisfied.

Then, u and v are monotone on each line in the e_n -direction, i.e., for any $x \in \Omega$, either $u_n(x+te_n) \geq 0$ or $u_n(x+te_n) \leq 0$, and either $v_n(x+te_n) \geq 0$ or $v_n(x+te_n) \leq 0$, for any $t \in \mathbb{R}$.

Remark 1.6. We notice that if a continuous function w is monotone on each line in \mathbb{R}^n , then it is one-dimensional, that is there exist a function $w_0 : \mathbb{R} \rightarrow \mathbb{R}$ and a unit direction $\omega \in \mathbb{S}^{n-1}$ such that $w(x) = w_0(\omega \cdot x)$ (see Section 9 in [18] for a proof of this fact).

1.4. A monotonicity result for stable solutions. The second result that we state concerns stable critical points of the energy instead of e_n -minimizers. We give a definition of the stability condition that involves the second variation of the functional \mathcal{E} for deformations of the solution both in the e_n -direction and in the vertical e_{n+1} -direction. Precisely, we give the following:

Definition 1.7. Given a function w , a piecewise Lipschitz deformation of w in the $\{e_n, e_{n+1}\}$ -directions is a function \tilde{w} defined as

$$\tilde{w} = \bar{w} + \phi,$$

where $\bar{w} \in D_R^\delta(w)$ and ϕ is a Lipschitz function with compact support in $\Omega \cap B_R$, and $\|\phi\|_{C^{0,1}(\Omega)} \leq \delta$. In this case we write $w \in \mathcal{D}_R^\delta(w)$.

Definition 1.8. We say that (u, v) is an $\{e_n, e_{n+1}\}$ -stable solution of \mathcal{E} if for every $R > 0$ and $\epsilon > 0$ there exists $\delta > 0$ depending on R, ϵ, u and v such that, for any $t \in (0, \delta)$, we have that $\mathcal{E}_R(u, v)$ is finite and

$$(1.6) \quad \mathcal{E}_R(\tilde{u}, \tilde{v}) - \mathcal{E}_R(u, v) \geq -\epsilon t^2, \quad \text{for any } \tilde{u} \in \mathcal{D}_R^t(u) \text{ and any } \tilde{v} \in \mathcal{D}_R^t(v).$$

Remark 1.9. As we shall see later on (see Lemma 2.1), this notion of $\{e_n, e_{n+1}\}$ -stable solution shares intimate links with the classical notion of stability. This actually follows from the same arguments as in the case of a single equation (see [18]). Actually, we can infer directly from the definition that

- (1) (u, v) is a classical minimizer of $\mathcal{E} \Rightarrow (u, v)$ is $\{e_n, e_{n+1}\}$ -stable for \mathcal{E} ;
- (2) (u, v) is $\{e_n, e_{n+1}\}$ -stable for $\mathcal{E} \Rightarrow (u, v)$ is a critical point for \mathcal{E} .

This is because we allow perturbation in the e_{n+1} -direction.

Next, we state our monotonicity result for $\{e_n, e_{n+1}\}$ -stable solutions.

Theorem 1.10. Let $F \in C^{3,\alpha}(\mathbb{R}^{2n} \times \mathbb{R}^2 \times V)$ satisfy (1.4) and (u, v) be such that

$$\begin{aligned} & \text{either } u, v \in C^{0,1}(\Omega) \text{ are convex} \\ & \text{or } u, v \in C^{1,1}(\Omega). \end{aligned}$$

Moreover, suppose that (u, v) is an $\{e_n, e_{n+1}\}$ -stable solution of \mathcal{E} , and that the growth condition (1.5) holds true.

Then, u and v are monotone in the e_n -direction, i.e. either $u_n \geq 0$ or $u_n \leq 0$ and either $v_n \geq 0$ or $v_n \leq 0$ in Ω .

By Theorem 1.10 and Remark 1.6 one obtains that an $\{e_n, e_{n+1}\}$ -stable solution of \mathcal{E} is one-dimensional, in the sense that there exist $\bar{u}, \bar{v} : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega_u, \omega_v \in \mathbb{S}^{n-1}$ such that $(u(x), v(x)) = (\bar{u}(\omega_u \cdot x), \bar{v}(\omega_v \cdot x))$. We remark that, in general, it is not possible to conclude that u and v have the same direction of monotonicity. It is not true, for instance, for an uncoupled system of PDEs, such as

$$\begin{cases} \Delta u = 0, \\ \Delta v = 0. \end{cases}$$

See also Remark 1.4 in [9] for a discussion on this fact.

On the other hand, there are cases in which it is possible to obtain that u and v have the same direction of monotonicity, even for quite general systems, see e.g. [9] and [10].

We stress that the aim of this paper is to generalize to systems a method to obtain monotonicity properties that adapts to quite general settings.

1.5. Organization of the paper. In Section 2 we show that, under suitable assumptions, the notion of $\{e_n, e_{n+1}\}$ -stability is equivalent to the notion of classical stability. In Section 3 we perform a local analysis and, in Section 4, we estimate the energy of the perturbation at infinity, collecting the basic facts in order to prove Theorems 1.5 and 1.10 in Sections 5 and 6, respectively. Finally, in Section 7 we investigate some applications of our results in some concrete cases.

2. STABILITY PROPERTY

The following lemma justifies Remark 1.9.

Lemma 2.1. *If $\Omega = \mathbb{R}^n$, $F \in C^2$ and $u, v \in C^2(\mathbb{R}^n)$, then (u, v) is a classical stable solution for the energy functional \mathcal{E} if, and only if, (u, v) is an $\{e_n, e_{n+1}\}$ -stable solution of \mathcal{E} according to Definition 1.8.*

Proof. Suppose first that (u, v) , with $u, v \in C^2(\mathbb{R}^n)$, is a classical stable solution. That is, by definition,

$$(2.1) \quad \liminf_{t \rightarrow 0} \frac{\mathcal{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathcal{E}_R(u, v)}{t^2} \geq 0 \quad \text{for any } \varphi^1, \varphi^2 \in C_0^{0,1}(B_R).$$

In addition, setting

$$(2.2) \quad \Delta_{\varphi^1, \varphi^2}^1 F := F(\nabla u + t\nabla\varphi^1, \nabla v + t\nabla\varphi^2, u + t\varphi^1, v + t\varphi^2, x') - F(\nabla u, \nabla v, u, v, x'),$$

we have

$$(2.3) \quad \Delta_{\varphi^1, \varphi^2}^1 F = tH(\varphi^1, \varphi^2) + \frac{t^2}{2}L(\varphi^1, \varphi^2) + o(t^2),$$

where

$$\begin{aligned} H(\varphi^1, \varphi^2) &:= F_{(p_1)_i} \varphi_i^1 + F_{z_1} \varphi^1 + F_{(p_2)_i} \varphi_i^2 + F_{z_2} \varphi^2 \\ \text{and } L(\varphi^1, \varphi^2) &:= F_{(p_1)_i(p_1)_j} \varphi_i^1 \varphi_j^1 + F_{z_1 z_1} (\varphi^1)^2 + 2F_{(p_1)_i z_1} \varphi_i^1 \varphi^1 \\ &\quad + F_{(p_2)_i(p_2)_j} \varphi_i^2 \varphi_j^2 + F_{z_2 z_2} (\varphi^2)^2 + 2F_{(p_2)_i z_2} \varphi_i^2 \varphi^2 \\ &\quad + F_{(p_1)_i(p_2)_j} \varphi_i^1 \varphi_j^2 + 2F_{z_2 z_1} \varphi^1 \varphi^2 + 2F_{(p_2)_i z_1} \varphi_i^2 \varphi^2 \\ &\quad + F_{(p_2)_i(p_1)_j} \varphi_i^2 \varphi_j^1 + 2F_{(p_1)_i z_2} \varphi_i^2 \varphi^1 \end{aligned}$$

and the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$.

Now we observe that, since (u, v) is a critical point of \mathcal{E} ,

$$\int_{B_R} H(\varphi^1, \varphi^2) = 0.$$

So integrating (2.3) and recalling (2.2), we obtain that

$$(2.4) \quad \mathcal{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathcal{E}_R(u, v) = \frac{t^2}{2} \int_{B_R} L(\varphi^1, \varphi^2) dx + o(t^2).$$

Dividing by t^2 and recalling (2.1) we find that

$$\int_{B_R} L(\varphi^1, \varphi^2) dx \geq 0.$$

Thus, (2.4) yields

$$(2.5) \quad \mathcal{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathcal{E}_R(u, v) \geq o(t^2).$$

Given $\tilde{u} \in \mathcal{D}_R^t(u)$ and $\tilde{v} \in \mathcal{D}_R^t(v)$ we may choose $\varphi^1 := \frac{\tilde{u}-u}{t}$ and $\varphi^2 := \frac{\tilde{v}-v}{t}$. By construction, the Lipschitz norm of φ^1 and φ^2 is bounded by a quantity that does not depend on t . Therefore (2.5) implies (1.6), and so it follows that (u, v) is $\{e_n, e_{n+1}\}$ -stable.

Reciprocally, let (u, v) be an $\{e_n, e_{n+1}\}$ -stable solution of \mathcal{E} (and so, a critical point). In this case, we choose $\tilde{u} := u + t\varphi^1$ and $\tilde{v} := v + t\varphi^2$ in (1.6). Then, taking ϵ arbitrarily small, we prove (2.1). Therefore (u, v) is a stable solution of \mathcal{E} . This completes the proof. \square

3. LOCAL ANALYSIS

In this section we will perform the local perturbation of a couple (u, v) . More precisely, we will show that we can perturb the couples

$$(\max\{u(x), u(x) + te_n\}, v(x)) \quad \text{and} \quad (u(x), \max\{v(x), v(x) + te_n\})$$

in such a way that the energy of the “perturbed couples” decreases.

We first show that if one of the two elements of a couple (u, v) is a maximum of two functions that form an angle at an intersection point, then it cannot be an e_n -minimizer for \mathcal{E} .

Lemma 3.1. *Suppose that $0 \in \Omega$, and that u, u^1, v, v^1 are C^1 functions which satisfy*

$$(3.1) \quad u(0) = u^1(0), \quad u_n^1(0) < 0 < u_n(0),$$

and

$$v(0) = v^1(0), \quad v_n(0) < 0 < v_n^1(0).$$

Then, the couples $g_1 := (\max\{u, u^1\}, v)$ and $g_2 := (u, \max\{v, v^1\})$ are not e_n -minimizers for \mathcal{E} in any ball B_η with $\eta > 0$.

Proof. We prove the lemma for the couple g_1 , the proof for g_2 being similar with obvious modifications.

We argue by contradiction and we assume that g_1 is an e_n -minimizer in a ball B_η , for some $\eta > 0$. We define $F_0(p_1, p_2) := F(p_1, p_2, 0, 0, 0)$. Moreover, we observe that if we subtract a linear functional from F then this does not affect the minimality. Indeed, we can consider

$$\tilde{F}(p_1, p_2, z_1, z_2, x) := F(p_1, p_2, z_1, z_2, x) - p_0 \cdot p_1,$$

then, if $\tilde{\mathcal{E}}$ is the associated energy functional, we have

$$\mathcal{E}_R(u, v) - \tilde{\mathcal{E}}_R(u, v) = \int_{B_R} p_0 \cdot \nabla u \, dx = \int_{\partial B_R} p_0 u \cdot \nu \, d\sigma.$$

This means that the difference between $\mathcal{E}_R(u, v)$ and $\tilde{\mathcal{E}}_R(u, v)$ is a term which depends only on the boundary value of u . Therefore, if (u, v) is an e_n -minimizer for \mathcal{E} , then it is an e_n -minimizer for $\tilde{\mathcal{E}}$ as well. From this we deduce that we may assume

$$(3.2) \quad F_0(\nabla u(0), \nabla v(0)) = F_0(\nabla u^1(0), \nabla v(0)).$$

Moreover, we can suppose that $u(0) = u^1(0) = 0$; for this, it is sufficient to translate F in the variable z_1 .

Now, for small $r > 0$, we define the rescaled functions

$$u_r(x) := r^{-1}u(rx), \quad u_r^1(x) := r^{-1}u^1(rx) \quad \text{and} \quad v_r(x) := r^{-1}v(rx).$$

We also set $g_r := (\max\{u_r, u_r^1\}, v_r)$. If

$$F_r(p_1, p_2, z_1, z_2, x) := F(p_1, p_2, rz_1, rz_2, rx)$$

is the rescaled functional, we have that g_r is an e_n -minimizer for F_r in $B_{\eta/r}$. Sending $r \rightarrow 0^+$, we obtain the following limits, which are uniform on compact sets:

$$(3.3) \quad \begin{aligned} F_r &\rightarrow F_0(p_1, p_2), \\ u_r(x) &\rightarrow u_0(x) := \nabla u(0) \cdot x, & \nabla u_r &\rightarrow \nabla u_0, \\ u_r^1(x) &\rightarrow u_0^1(x) := \nabla u^1(0) \cdot x, & \nabla u_r^1 &\rightarrow \nabla u_0^1, \\ v_r(x) &\rightarrow v_0(x) := \nabla v(0) \cdot x, & \nabla v_r &\rightarrow \nabla v_0. \end{aligned}$$

Now, we set $f_0 := \max\{u_0, u_0^1\}$ and we consider the couple $g_0 = (f_0, v_0)$. The fact that F is strictly convex in the variable p_1 implies that

$$(3.4) \quad g_0 \text{ is not a minimizer for } F_0.$$

To see this, given the function

$$\beta_R(x') := c\delta \max\{0, |x'| - R\},$$

with R large and c, δ small, we define

$$(3.5) \quad h_0 := 1 + \alpha u_0 + (1 - \alpha)u_0^1 - \beta_R(x'),$$

where $\alpha \in (0, 1)$ is small. Then, we have that

$$(3.6) \quad \text{the couple } (\max\{f_0, h_0\}, v_0) \text{ agrees with } g_0 \text{ outside the ball } B_{R+R_1}.$$

On the other hand, in B_R , considering h_0 , we cut the graphs of two transversal linear functions by a single one. Therefore, if we take R sufficiently large,

$$(3.7) \quad \text{the couple } (\max\{f_0, h_0\}, v_0) \text{ has lower energy for } F_0 \text{ than } g_0.$$

To see this, we observe that, by definition, ∇f_0 coincides with either ∇u_0 or ∇u_0^1 , and so from (3.2) we have that

$$F_0(\nabla u_0, \nabla v_0) = F_0(\nabla u_0^1, \nabla v_0) = F_0(\nabla f_0, \nabla v_0).$$

Therefore, by the strict convexity of F_0 with respect to the first variable, we have that, fixed α and δ ,

$$(3.8) \quad \begin{aligned} F_0(\nabla h_0, \nabla v_0) &= F_0((\alpha \nabla u_0 + (1 - \alpha) \nabla u_0^1, \nabla v_0)) \\ &\leq \alpha F_0(\nabla u_0, \nabla v_0) + (1 - \alpha) F_0(\nabla u_0^1, \nabla v_0) - \eta \\ &= F_0(\nabla f_0, \nabla v_0) - \eta \end{aligned}$$

in $B_R \cap \{h_0 > f_0\}$, for some $\eta > 0$. On the other hand, the set $\{h_0 > f_0\}$ is contained in a strip in the e_n -direction. This and (3.6) imply that, when we integrate (3.8) in B_R , then the energy of $(\max\{f_0, h_0\}, v_0)$ is less than the energy of g_0 minus a term of order $\underline{C}\eta R^{n-1}$, whereas, when we integrate in B_{R+R_1} , the energy of $(\max\{f_0, h_0\}, v_0)$ is less than the energy of g_0 minus a term of order $\overline{C}\eta R^{n-2}$. All in all, if R is large enough we obtain (3.7). In turn, this implies (3.4).

Now we set

$$(3.9) \quad f_r := \max\{u_r, u_r^1\}.$$

Since the convergence in (3.3) is uniform, we have that

$$h_r := (\max\{f_r, h_0\}, v_r)$$

has lower energy for F_r than g_r , as well.

Hence, we can scale back and obtain that $h_*(x) := rh_r(x/r)$ has lower energy for F in $B_{r(R+R_1)} \subseteq B_\eta$ than the one of g_1 .

In order to get a contradiction, it remains to prove that h_* is an allowed perturbation of g_1 (according to Definition 1.2). Notice that this is equivalent to check that h_r is an allowed perturbation of g_r . That is, recalling (3.9), we have to prove that

$$(3.10) \quad \begin{aligned} &\max\{f_r, h_0\} \text{ is a piecewise Lipschitz domain deformation of } f_r \\ &\text{with the Lipschitz norm bounded by } \delta. \end{aligned}$$

To do this, we recall the uniform convergence of u_r and u_r^1 to u_0 and u_0^1 , respectively, given by (3.3), and the definition of h_0 given in (3.5) to obtain that

$$(3.11) \quad h_0(x) = 1 + \alpha u_r(x) + (1 - \alpha) u_r^1(x) - \beta_R(x') + \omega_r(x),$$

where $\omega_r \rightarrow 0$ as $r \rightarrow 0^+$ locally uniformly, together with its derivatives.

Now we notice that our hypothesis in (3.1) gives that $\nabla u_0^1 \cdot e_n < 0 < \nabla u_0 \cdot e_n$. This, together with the uniform convergence in (3.3), implies that we can apply the Implicit Function Theorem, and we have that the part of the graph of $\max\{f_r, h_0\}$ where $h_0 > f_r$ can be obtained from u_r by a Lipschitz domain deformation with Lipschitz norm less than δ , provided that we take α sufficiently small. Indeed, fixed x' (and so looking at the 1-dimensional problem in the last variable only) and recalling (3.11), we obtain that

$$u_r(x', x_n + \varphi(x)) = h_0(x) = 1 + \alpha u_r(x) + (1 - \alpha) u_r^1(x) - \beta_R(x') + \omega_r(x),$$

thanks to the Implicit Function Theorem in 1-dimension. Furthermore, if α and r are sufficiently small, the perturbation function φ has norm bounded by $\delta/2$. This shows (3.10) and finishes the proof of Lemma 3.1. \square

Now we deal with perturbations of the couples

$$(\max\{u(x), u(x) + te_n\}, v(x)) \quad \text{and} \quad (u(x), \max\{v(x), v(x) + te_n\})$$

with lower energy.

Lemma 3.2. *Let $u, v \in C^2(\Omega)$ be such that (u, v) is a critical point for the functional \mathcal{E} in a neighborhood of the origin, and let $F \in C^2$ in a neighborhood of $(\nabla u(0), \nabla v(0), u(0), v(0), 0)$. Suppose that*

$$u_n(0) = 0, \quad \nabla u_n(0) \neq 0,$$

and set

$$(3.12) \quad w^1(x) := \max\{u(x), u(x + te_n)\}.$$

Then, for every $\eta > 0$ there exists a function ψ^1 which is Lipschitz and has compact support in B_η such that, for any small t ,

$$\mathcal{E}_\eta(w^1 + t\psi^1, v) - \mathcal{E}_\eta(w^1, v) \leq -ct^2,$$

where $c > 0$ is a small constant that depends on F , η and u .

Remark 3.3. We point out that we have an analogous result if we consider the function v instead of u . Precisely, if we assume that

$$v_n(0) = 0, \quad \nabla v_n(0) \neq 0,$$

and we consider

$$w^2(x) := \max\{v(x), v(x + te_n)\},$$

then, for every $\eta > 0$ there exists a function ψ^2 which is Lipschitz and has compact support in B_η such that, for any small t ,

$$\mathcal{E}_\eta(u, w^2 + t\psi^2) - \mathcal{E}_\eta(u, w^2) \leq -ct^2,$$

where $c > 0$ is a small constant depending on F , η and v .

Proof of Lemma 3.2. We define

$$(3.13) \quad u^1(x) := \frac{u(x + te_n) - u(x)}{t}$$

and we observe that

$$(3.14) \quad \|u^1 - u_n\|_{C^{0,1}(B_\eta)} = o(1) \quad \text{as } t \rightarrow 0.$$

We consider a Lipschitz function g_1 and, using the fact that $F \in C^2$ in the variables p_1 and z_1 , we compute

$$\begin{aligned} & F(\nabla u + t\nabla g_1, \nabla v, u + tg_1, v, x') \\ &= F(\nabla u, \nabla v, u, v, x') + t(F_{p_1}\nabla g_1 + F_{z_1}g_1) \\ & \quad + t^2\left((\nabla g_1)^T F_{p_1 p_1}\nabla g_1 + F_{z_1 z_1}g_1^2 + 2g_1 F_{p_1 z_1} \cdot \nabla g_1\right) + o(t^2), \end{aligned}$$

where the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$ and the constant in the error term $o(t^2)$ depends on u , F and $\|g_1\|_{C^{0,1}(B_\eta)}$. Hence, we obtain

$$\mathcal{E}_\eta(u + tg_1, v) = \mathcal{E}_\eta(u, v) + tL(g_1) + t^2Q(g_1) + o(t^2),$$

with

$$L(g_1) := \int_{B_\eta} (F_{p_1} \cdot \nabla g_1 + F_{z_1} g_1) dx$$

and

$$\begin{aligned} Q(g_1) &:= \int_{B_\eta} G(\nabla g_1, g_1, x) dx \\ &= \int_{B_\eta} \left((\nabla g_1)^T F_{p_1 p_1} \nabla g_1 + F_{z_1 z_1} g_1^2 + 2g_1 F_{p_1 z_1} \cdot \nabla g_1 \right) dx. \end{aligned}$$

Now, we take a Lipschitz function ψ^1 with compact support in B_η , and we use the fact that (u, v) is a critical point for \mathcal{E} to obtain

$$(3.15) \quad \mathcal{E}_\eta(u + tu_+^1 + t\psi^1, v) - \mathcal{E}_\eta(u + tu_+^1, v) = t^2 (Q(u_+^1 + \psi^1) - Q(u_+^1)) + o(t^2).$$

Using (3.12) and (3.13), we can write

$$(3.16) \quad w^1 = u + tu_+^1.$$

Now we claim that, for η sufficiently small,

$$(3.17) \quad Q(u_+^1) - Q((u_n)_+) = o(1) \quad \text{and} \quad Q(u_+^1 + \psi^1) - Q((u_n)_+ + \psi^1) = o(1), \quad \text{as } t \rightarrow 0.$$

We focus on the first equality in (3.17), since the second is similar. To prove it, fixed $\mu > 0$, we define

$$\mathcal{B}_\mu^1 := B_\eta \cap \{|u_n| \leq \mu\} \quad \text{and} \quad \mathcal{B}_\mu^2 := B_\eta \cap \{|u_n| > \mu\}.$$

Notice that (3.14) implies that

$$(3.18) \quad \lim_{t \rightarrow 0} \|u_+^1 - (u_n)_+\|_{C^{0,1}(\mathcal{B}_\mu^2)} = \lim_{t \rightarrow 0} \|u^1 - u_n\|_{C^{0,1}(\mathcal{B}_\mu^2)} = 0.$$

As for the contribution coming from \mathcal{B}_μ^1 , we observe that, since $\nabla u_n(0) \neq 0$, the measure of \mathcal{B}_μ^1 is at most of order of μ . This, together with (3.18), gives that

$$\lim_{t \rightarrow 0} |Q(u_+^1) - Q((u_n)_+)| \leq C\mu,$$

for some $C > 0$. Since μ can be taken arbitrarily small, this implies (3.17).

Formula (3.17) means that, for η sufficiently small, we can write $(u_n)_+$ instead of u_+^1 in the right hand side of (3.15). Hence, recalling also (3.16), we have that

$$(3.19) \quad \mathcal{E}_\eta(w^1 + t\psi^1, v) - \mathcal{E}_\eta(w^1, v) = t^2 (Q((u_n)_+ + \psi^1) - Q((u_n)_+)) + o(t^2).$$

Now, we notice that u_n , 0 and G satisfy the hypotheses of Remark 4.3 in [18], and therefore $(u_n)_+$ is not a minimizer of Q . So we can take the function ψ^1 in such a way that

$$Q((u_n)_+ + \psi^1) \leq Q((u_n)_+) - c$$

for some small c , which may depend on u , F and η . This together with (3.19) gives that, for small t ,

$$\mathcal{E}_\eta(w^1 + t\psi^1, v) - \mathcal{E}_\eta(w^1, v) \leq -ct^2$$

and this concludes the proof. \square

Now we show that, under an additional regularity hypothesis on F , the non-degeneracy condition $\nabla u_n \neq 0$ in Lemma 3.2 is satisfied. For this we use the Hopf Lemma, by adapting the proof of Lemma 4.6 in [18] to the slightly more delicate case of a system of equations.

Lemma 3.4. *Assume that*

$$\begin{aligned} & \text{either } u, v \in C^{0,1}(\Omega) \text{ are convex} \\ & \text{or } u, v \in C^{1,1}(\Omega). \end{aligned}$$

Furthermore, assume that (u, v) is a critical point for \mathcal{E} and $F \in C^{3,\alpha}$ in a neighborhood of $(\nabla u(0), \nabla v(0), u(0), v(0), 0)$.

Then, u and v are of class $C^{3,\alpha}$ in a neighborhood of 0.

If, in addition, $u_n(0) = 0$ (resp. $v_n(0) = 0$) and u_n (resp. v_n) does not vanish identically in a neighborhood V_0 of 0, then there exists a point $x_0 \in V_0$ (resp. $x_1 \in V_0$) such that

$$u_n(x_0) = 0, \nabla u_n(x_0) \neq 0 \text{ and } v_n(x_1) = 0, \nabla v_n(x_1) \neq 0$$

Proof. Since (u, v) is a critical point for \mathcal{E} it satisfies the elliptic system of equation

$$\begin{cases} G^1(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x') := \operatorname{div}_x F_{p_1}(M) - F_{z_1}(M) = 0 \\ G^2(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x') := \operatorname{div}_x F_{p_2}(M) - F_{z_2}(M) = 0 \end{cases}$$

where $M := (\nabla u, \nabla v, u, v, x')$. Consider the first equation of the above system, with v fixed. If $v \in C^{0,1}(\Omega)$ is convex, then $\nabla v \in L^\infty(\Omega)^n$ and $\nabla^2 v \in L^\infty(\Omega)^{n \times n}$ exist almost everywhere (this follows from Rademacher Theorem and Alexandrov Theorem). On the other hand, if $v \in C^{1,1}(\Omega)$, then $\nabla^2 v \in L^\infty(\Omega)^{n \times n}$ also exists almost everywhere (this follows from Rademacher Theorem). Thus, at fixed v , the first equation (in the variable $u = u_v$) is satisfied in the classical sense and the corresponding solution u_v belongs to $C^{3,\alpha}(\Omega)$ (this follows from Theorem 2.1 in [22], the Schauder estimates and the fact that $F \in C^{3,\alpha}$). In like manner, for any fixed u , the solution v_u of the second¹ equation belongs to $C^{3,\alpha}(\Omega)$. Therefore, $(u, v) \in C^{3,\alpha}(\Omega) \times C^{3,\alpha}(\Omega)$. This shows the first claim of Lemma 3.4.

Now we focus on the second claim of Lemma 3.4. To prove it, we observe that

$$(3.20) \quad G^1 = G^1(q_1, q_2, p_1, p_2, z_1, z_2, x') \in C^{1,\alpha}.$$

Hence, differentiating in the e_n -direction we see that $w := u_n \in C^{2,\alpha}(\Omega)$ satisfies the linearized equation (in the viscosity sense)

$$(3.21) \quad G^1_{(q_1)_{ij}} w_{ij} + G^1_{(p_1)_i} w_i + G^1_{z_1} w = -G^1_{(q_2)_{ij}} (v_n)_{ij} - G^1_{(p_2)_i} (v_n)_i - G^1_{z_2} v_n.$$

Here, the derivatives of G^1 are evaluated at $(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x')$. For the convenience of the reader, we rewrite (3.21) as

$$L^1 w = f_1,$$

where

$$\begin{aligned} L^1 w &:= G^1_{(q_1)_{ij}} w_{ij} + G^1_{(p_1)_i} w_i + G^1_{z_1} w \\ \text{and } f_1 &:= -G^1_{(q_2)_{ij}} (v_n)_{ij} - G^1_{(p_2)_i} (v_n)_i - G^1_{z_2} v_n. \end{aligned}$$

Moreover, we set

$$\begin{aligned} P &:= \{x \in V_0 : f_1(x) > 0\}, \\ E &:= \{x \in V_0 : f_1(x) = 0\} \\ \text{and } N &:= \{x \in V_0 : f_1(x) < 0\}. \end{aligned}$$

¹We may actually consider indifferently the first or the second equation, as only one equation is needed.

In addition, denote by $(P_j)_{j \in J_1}$ (resp. $(E_j)_{j \in J_2}$) the connected components of P (resp. E). This makes sense since $f_1 \in C^{0,\alpha}(V_0)$ (recall (3.20) and the fact that v is $C^{3,\alpha}$ in V_0).

Note that if $P = N = \emptyset$ then $f_1 \equiv 0$, and so the conclusion follows as in Lemma 4.6 in [18]. Thus we may suppose now that either P or N are nonempty. Moreover, we can assume that P is nonempty, otherwise it is enough to replace w by $-w$.

Also, we observe that

$$(3.22) \quad \begin{aligned} & \text{if there exists a compact set } \Sigma \Subset V_0 \text{ such that } w|_\Sigma \equiv 0, \\ & \text{then necessarily } \Sigma \subset E. \end{aligned}$$

Now we take² $x_0 \in \{w = 0\} \cap P$. Then there exists $j_1 \in J_1$ such that $x_0 \in P_{j_1}$. It follows from (3.22) and the continuity of f_1 that w cannot vanish identically in P_{j_1} . Thus, we can apply the Hopf Lemma to w at the point x_0 , which admits a tangent ball from either $\{w < 0\} \cap P_{j_1}$ or $\{w > 0\} \cap P_{j_2}$, and the conclusion follows.

If $x_0 \in \{w = 0\} \cap N$, we replace w with $-w$ and we reason as above.

If $x_0 \in \{w = 0\} \cap E$, we let E_{j_2} , for some $j_2 \in J_2$, be the connected component of E such that $x_0 \in E_{j_2}$. By the continuity of f_1 , there exists $j_1 \in J_1$ such that $K := E_{j_2} \cup P_{j_1}$ is connected (up to exchanging w with $-w$). Again, (3.22) implies that w cannot vanish identically in K , and so we can apply the Hopf Lemma to w at x_0 , which admits a tangent ball from either $\{w < 0\} \cap K$ or $\{w > 0\} \cap K$.

The same arguments apply to v_n . This completes the proof of Lemma 3.4. \square

Remark 3.5. The above result states that the non-degeneracy hypothesis $\nabla u_n \neq 0$ of Lemma 3.2 is always satisfied under a slightly stronger hypothesis on the potential F . On the other hand, notice that we obtain a $C^{3,\alpha}$ regularity for u and v near 0, even though we asked only for, say, a $C^{1,1}$ regularity. Thus, Lemma 3.2 is consistent.

4. PERTURBATIONS AT INFINITY

In this section we modify the couples

$$(\max\{u(x), u(x) + te_n\}, v(x)) \quad \text{and} \quad (u(x), \max\{v(x), v(x) + te_n\})$$

at infinity in such a way that they become compact perturbations of the couple (u, v) .

For this, for any $R > 1$ we define the function $\varphi_R : \mathbb{R} \rightarrow \mathbb{R}$, which is Lipschitz, even and with compact support, as

$$(4.1) \quad \varphi_R(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq \sqrt{R}, \\ 2 \frac{\log R - \log s}{\log R} & \text{if } \sqrt{R} < s < R, \\ 0 & \text{if } s > R. \end{cases}$$

We have that

$$(4.2) \quad \varphi'_R(s) := \begin{cases} 0 & \text{if } s \in (0, \sqrt{R}) \cup (R, +\infty), \\ \frac{-2}{s \log R} & \text{if } s \in (\sqrt{R}, R). \end{cases}$$

²We use here the short notation $\{w = 0\} = \{x \in V_0 : w(x) = 0\}$, and similarly for $\{w < 0\}$ and $\{w > 0\}$.

Also, for any $t \in (0, \sqrt{R}/4]$, we consider the following bi-Lipschitz change of coordinates:

$$(4.3) \quad x \mapsto y(x) := x + t\varphi_R(|x|)e_n.$$

In these new coordinates, we define the functions $u_{R,t}^+$ and $v_{R,t}^+$ as

$$u_{R,t}^+(y) := u(x) \quad \text{and} \quad v_{R,t}^+(y) = v(x).$$

We observe that $u_{R,t}^+(x)$ and $v_{R,t}^+(x)$ coincide with $u(x-te_n)$ and $v(x-te_n)$ respectively in $B_{\sqrt{R}/2}$ and with $u(x)$ and $v(x)$ respectively outside B_R .

We also define $u_{R,t}^-$ and $v_{R,t}^-$ by replacing t with $-t$ in (4.3).

Now, we want to obtain an estimate of $\mathcal{E}_R(u_{R,t}^+, v_{R,t}^+)$ and $\mathcal{E}_R(u_{R,t}^-, v_{R,t}^-)$ in terms of $\mathcal{E}_R(u, v)$. For this, we notice that

$$D_x y = I + A,$$

where

$$A(x) := t\varphi'_R(|x|) \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1}{|x|} & \frac{x_2}{|x|} & \dots & \frac{x_n}{|x|} \end{pmatrix}$$

and $\|A\| \leq t|\varphi'_R(|x|)| \ll 1$. Moreover,

$$D_y x = (I + A)^{-1} = I - \frac{1}{1 + \text{tr} A} A.$$

Furthermore, the following relations hold:

$$\nabla_y u_{R,t}^+ = \nabla_x u D_y x, \quad \nabla_y v_{R,t}^+ = \nabla_x v D_y x \quad \text{and} \quad dy = (1 + \text{tr} A) dx.$$

Hence, we have

$$\begin{aligned} & \int_{\Omega \cap B_R} F(\nabla_y u_{R,t}^+, \nabla_y v_{R,t}^+, u_{R,t}^+, v_{R,t}^+, y') dy \\ &= \int_{\Omega \cap B_R} F\left(\nabla_x u \left(I - \frac{1}{1 + \text{tr} A} A\right), \nabla_x v \left(I - \frac{1}{1 + \text{tr} A} A\right), u, v, x'\right) (1 + \text{tr} A) dx. \end{aligned}$$

Now we use the hypothesis (1.4) for F to bound the right hand side from above: more precisely, since $|(p_1 A)| \leq |p_1 \cdot e_n|/4$ and $|(p_2 A)| \leq |p_2 \cdot e_n|/4$, we have that

$$\begin{aligned} & F\left(p_1 \left(I - \frac{1}{1 + \text{tr} A} A\right), p_2 \left(I - \frac{1}{1 + \text{tr} A} A\right), z_1, z_2, x'\right) (1 + \text{tr} A) \\ &\leq F(p_1, p_2, z_1, z_2, x') (1 + \text{tr} A) \\ &\quad - F_{p_1}(p_1, p_2, z_1, z_2, x') \cdot (p_1 A) - F_{p_2}(p_1, p_2, z_1, z_2, x') \cdot (p_2 A) \\ &\quad + C \left[|F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')| |p_1 A|^2 + |F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')| |p_2 A|^2 \right. \\ &\quad \left. + 2 |F_{p_1 p_2}(p_1, p_2, z_1, z_2, x')| |p_1 A| |p_2 A| \right]. \end{aligned}$$

If we consider $u_{R,t}^-$ and $v_{R,t}^-$, we can write the same inequality with A replaced by $-A$. Therefore, we obtain

$$(4.4) \quad \begin{aligned} & \mathcal{E}_R(u_{R,t}^+, v_{R,t}^+) + \mathcal{E}_R(u_{R,t}^-, v_{R,t}^-) - 2\mathcal{E}_R(u, v) \\ & \leq C \int_{\Omega \cap B_R} \left[|F_{p_1 p_1}| |\nabla u|^2 |A|^2 + |F_{p_2 p_2}| |\nabla v|^2 |A|^2 + 2 |F_{p_1 p_2}| |\nabla u| |\nabla v| |A|^2 \right] dx \\ & \leq C \frac{t^2}{(\log R)^2} \int_{\Omega \cap (B_R \setminus B_{\sqrt{R}})} \frac{|F_{p_1 p_1}| |\nabla u|^2 + |F_{p_2 p_2}| |\nabla v|^2 + |F_{p_1 p_2}| |\nabla u| |\nabla v|}{|x|^2} dx, \end{aligned}$$

where the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$. Now we set

$$h(r) := \int_{\Omega \cap B_r} (|F_{p_1 p_1}| |\nabla u|^2 + |F_{p_2 p_2}| |\nabla v|^2 + |F_{p_1 p_2}| |\nabla u| |\nabla v|) dx.$$

Recalling (1.5), we have that $h(r) \leq Cr^2$. Also, we see that

$$(4.5) \quad \int_{\sqrt{R}}^R \frac{h'(r)}{r^2} dr \leq \frac{h(R)}{R^2} + 2 \int_{\sqrt{R}}^R \frac{h(r)}{r^3} dr \leq C \log R.$$

Passing to polar coordinates in the last integral in (4.4) and using (4.5) we get

$$\limsup_{R \rightarrow +\infty} \sup_{t \in (0, \sqrt{R}/4)} \frac{\mathcal{E}_R(u_{R,t}^+, v_{R,t}^+) + \mathcal{E}_R(u_{R,t}^-, v_{R,t}^-) - 2\mathcal{E}_R(u, v)}{t^2} \leq 0.$$

Remark 4.1. We point out that in a similar way, by perturbing only one of the element of the couple (u, v) , one can obtain

$$(4.6) \quad \limsup_{R \rightarrow +\infty} \sup_{t \in (0, \sqrt{R}/4)} \frac{\mathcal{E}_R(u_{R,t}^+, v) + \mathcal{E}_R(u_{R,t}^-, v) - 2\mathcal{E}_R(u, v)}{t^2} \leq 0$$

and

$$(4.7) \quad \limsup_{R \rightarrow +\infty} \sup_{t \in (0, \sqrt{R}/4)} \frac{\mathcal{E}_R(u, v_{R,t}^+) + \mathcal{E}_R(u, v_{R,t}^-) - 2\mathcal{E}_R(u, v)}{t^2} \leq 0.$$

We conclude this section recalling the following integral formulas:

$$(4.8) \quad \begin{aligned} & \mathcal{E}_R \left(\max \left\{ u_{R,t}^-, u \right\}, v \right) + \mathcal{E}_R \left(\min \left\{ u_{R,t}^-, u \right\}, v \right) = \mathcal{E}_R(u_{R,t}^-, v) + \mathcal{E}_R(u, v), \\ & \mathcal{E}_R \left(u, \max \left\{ v_{R,t}^-, v \right\} \right) + \mathcal{E}_R \left(u, \min \left\{ v_{R,t}^-, v \right\} \right) = \mathcal{E}_R(u, v_{R,t}^-) + \mathcal{E}_R(u, v). \end{aligned}$$

5. PROOF OF THEOREM 1.5

We recall the notation introduced at the beginning of Section 4, and we observe that, since (u, v) is an e_n -minimizer,

$$\mathcal{E}_R(u, v) \leq \mathcal{E}_R(u_{R,t}^+, v).$$

From this and (4.6) we obtain that

$$(5.1) \quad \lim_{R \rightarrow +\infty} \mathcal{E}_R(u_{R,t}^-, v) - \mathcal{E}_R(u, v) = 0,$$

at t fixed.

Using again the minimality of (u, v) , we have that

$$\mathcal{E}_R(u, v) \leq \mathcal{E}_R \left(\min \left\{ u_{R,t}^-, u \right\}, v \right),$$

which, together with the first relation in (4.8), implies that

$$(5.2) \quad \mathcal{E}_R \left(\max \left\{ u_{R,t}^-, u \right\}, v \right) - \mathcal{E}_R(u_{R,t}^-, v) = \mathcal{E}_R(u, v) - \mathcal{E}_R \left(\min \left\{ u_{R,t}^-, u \right\}, v \right) \leq 0.$$

Putting together (5.1) and (5.2) we obtain

$$(5.3) \quad \lim_{R \rightarrow +\infty} \mathcal{E}_R \left(\max \left\{ u_{R,t}^-, u \right\}, v \right) - \mathcal{E}_R(u, v) = 0.$$

We set

$$(5.4) \quad f_{R,t} := \max \left\{ u_{R,t}^-, u \right\}$$

and we observe that

$$f_{R,t} = \max \{ u(x), u(x + te_n) \}$$

and $f_{R,t} \in D_R^t(u)$.

Now, we argue by contradiction, assuming that $u \in C^1(\Omega)$ is not monotone on a line in the direction e_n . This implies that we can take $t > 0$ in such a way that $u(x)$ and $u(x + te_n)$ satisfy the hypotheses of Lemma 3.1, say at some point $x_0 \in \Omega$ (see Remark 4.2 in [18]). Therefore, we have that $g_{R,t} := (f_{R,t}, v)$ is not an e_n -minimizer for \mathcal{E} . Hence, in a neighborhood of x_0 , we can perturb $g_{R,t}$ into $\tilde{g}_{R,t}$ in such a way that

$$\mathcal{E}_R(\tilde{g}_{R,t}) \leq \mathcal{E}_R(g_{R,t}) - c$$

for some $c > 0$ which depends only on (u, v) . From the last inequality and (5.3) we reach a contradiction with the minimality of (u, v) as $R \rightarrow +\infty$.

If we assume that $v \in C^1(\Omega)$ is not monotone on a line in the e_n -direction, we get again a contradiction with the fact that (u, v) is an e_n -minimizer. Indeed we can repeat the same argument as above, using (4.7) and the second integral formula in (4.8). This concludes the proof of Theorem 1.5.

6. PROOF OF THEOREM 1.10

We recall the notation introduced at the beginning of Section 4. From (4.6) we deduce that, for $\epsilon > 0$, we can take R large such that

$$(6.1) \quad \mathcal{E}_R(u_{R,t}^+, v) + \mathcal{E}_R(u_{R,t}^-, v) - 2\mathcal{E}_R(u, v) \leq \epsilon t^2.$$

Moreover, we know that (u, v) is $\{e_n, e_{n+1}\}$ -stable, and so

$$(6.2) \quad \mathcal{E}_R(w_1, v) - \mathcal{E}_R(u, v) \geq -\epsilon t^2 \quad \text{for any } w_1 \in \mathcal{D}_R^t(u),$$

for every t sufficiently small (see Definition 1.8).

Now, recalling the definition of $f_{R,t}$ in (5.4), we use the first integral formula in (4.8) to obtain that

$$\begin{aligned} & \mathcal{E}_R(f_{R,t}, v) - \mathcal{E}_R(u, v) \\ &= \mathcal{E}_R(u_{R,t}^-, v) - \mathcal{E}_R \left(\min \left\{ u_{R,t}^-, u \right\}, v \right) \\ &= \mathcal{E}_R(u_{R,t}^-, v) + \mathcal{E}_R(u_{R,t}^+, v) - 2\mathcal{E}_R(u, v) + \mathcal{E}_R(u, v) \\ & \quad - \mathcal{E}_R \left(\min \left\{ u_{R,t}^-, u \right\}, v \right) + \mathcal{E}_R(u, v) - \mathcal{E}_R(u_{R,t}^+, v) \\ &\leq 3\epsilon t^2, \end{aligned}$$

where we have used (6.1) and (6.2).

By contradiction, we assume that u_n changes sign in Ω . Then, by Lemma 3.4, there exists a point $x_0 \in \Omega$ such that the hypotheses of Lemma 3.2 are satisfied in a neighborhood of x_0 . Hence, we have that we can perturb $f_{R,t}$ into $\tilde{f}_{R,t}$ near x_0 in such a way that

$$\mathcal{E}_R(\tilde{f}_{R,t}, v) \leq \mathcal{E}_R(f_{R,t}, v) - ct^2,$$

where $\tilde{f}_{R,t} \in \mathcal{D}_R^{Ct}(u)$, for some $c, C > 0$ which depend only on u . Then, we obtain

$$\mathcal{E}_R(\tilde{f}_{R,t}, v) \leq \mathcal{E}_R(u, v) + (3\epsilon - c)t^2,$$

which gives a contradiction with the stability inequality (1.6) if we take $\epsilon \ll c$.

If we assume that v_n changes sign in Ω , we reason in a similar way, using (4.7), the second integral formula in (4.8) and Remark 3.3 to reach the same contradiction. Therefore, either $u_n \geq 0$ or $u_n \leq 0$ and either $v_n \geq 0$ or $v_n \leq 0$ in Ω . This completes the proof of Theorem 1.10.

7. SOME APPLICATIONS

7.1. Two-stated mixture of Bose-Einstein condensate. Here we consider the following system

$$(7.1) \quad \begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ u, v > 0. \end{cases}$$

As already mentioned in the Introduction, the above system arises in the analysis of phase separation phenomena in binary mixtures of Bose-Einstein condensates with multiple states.

The energy associated to (7.1) is the following:

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2.$$

Under a suitable growth condition, we show here that stable solutions of (7.1) are one-dimensional, i.e. there exist $\bar{u}, \bar{v} : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega_u, \omega_v \in \mathbb{S}^{n-1}$ such that $(u(x), v(x)) = (\bar{u}(\omega_u \cdot x), \bar{v}(\omega_v \cdot x))$. Precisely:

Proposition 7.1. *Let $u, v \in C^2(\mathbb{R}^n)$. Suppose that (u, v) is a stable solution to (7.1), and that the following growth condition holds true: there exists a constant $C > 0$ such that, for R sufficiently large,*

$$(7.2) \quad \int_{B_R} |\nabla u|^2 + |\nabla v|^2 \leq CR^2.$$

Then (u, v) possesses one-dimensional symmetry.

Proof. We define

$$F(p_1, p_2, z_1, z_2, x') := \frac{1}{2} (|p_1|^2 + |p_2|^2 + z_1^2 z_2^2).$$

Notice that F is convex in p_1 and p_2 and satisfies (1.4). Moreover $F_{p_1 p_1} = 1 = F_{p_2 p_2}$ and $F_{p_1 p_2} = 0$, therefore the growth condition (1.5) is ensured by (7.2). Then, we can apply Theorem 1.10 (recall also Lemma 2.1) and we obtain that (u, v) is one-dimensional. \square

As a corollary, we obtain the one-dimensional symmetry in dimension 2 for stable solutions to (7.1) which have linear growth at infinity (see [2]).

Corollary 7.2. *Let $n = 2$ and let $u, v \in C^2(\mathbb{R}^2)$. Suppose that (u, v) is a stable solution to (7.1), and that the following growth condition holds true: there exists a constant $C > 0$ such that*

$$(7.3) \quad |u(x)| + |v(x)| \leq C(1 + |x|).$$

Then (u, v) possesses one-dimensional symmetry.

Proof. We observe that, by Theorem 5.1 in [24], if (u, v) satisfies (7.3) then $|\nabla u|$ and $|\nabla v|$ are bounded in \mathbb{R}^2 . Therefore, condition (7.2) is trivially satisfied, and so we get the desired result thanks to Proposition 7.1. \square

7.2. General systems with p -Laplacian type operators. Theorem 1.10 actually applies to a broader class of operators and nonlinearities. Indeed, with the notation introduced in [9] (see in particular pages 3474-3475 there), one can consider

$$(7.4) \quad \begin{cases} \operatorname{div}(a(|\nabla u|)\nabla u) = \tilde{F}_1(u, v), \\ \operatorname{div}(b(|\nabla v|)\nabla v) = \tilde{F}_2(u, v), \end{cases}$$

where \tilde{F} is a $C_{loc}^{1,1}$ function on \mathbb{R}^2 (it corresponds to the function F introduced in [9], here denoted as \tilde{F} to avoid confusion), and \tilde{F}_1, \tilde{F}_2 denote the derivatives of \tilde{F} with respect to the first and the second variable respectively.

Then, we have the following:

Proposition 7.3. *Let $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$ and $v \in C^1(\mathbb{R}^2) \cap C^2(\{\nabla v \neq 0\})$. Suppose that (u, v) is a stable solution to (7.4), and that conditions (B1) and (B2) in [13] are satisfied for a, b, A and B .*

Also, assume that the following growth condition holds true: there exists a constant $C > 0$ such that, for R sufficiently large,

$$(7.5) \quad \int_{B_R} \Lambda_2(|\nabla u|) + \Gamma_2(|\nabla v|) \leq CR^2.$$

where Λ_2 and Γ_2 are as in [9]. Then (u, v) possesses one-dimensional symmetry.

Proof. We define

$$F(p_1, p_2, z_1, z_2, x') := \Lambda_2(|p_1|) + \Gamma_2(|p_2|) + \tilde{F}(z_1, z_2),$$

and we verify that

$$\int_{\mathbb{R}^n} F(\nabla u, \nabla v, u, v, x') dx$$

satisfies the hypotheses needed to apply Theorem 1.10. Observe, first, that being stable for the above energy is the same as being stable for (7.4) (see Definition 1.2 in [9]). Now, recalling the notations used in [9], we derive

$$\begin{aligned} F_{(p_1)_i}(p_1, p_2, z_1, z_2, x') &= \lambda_2(|p_1|)_{(p_1)_i} = a(|p_1|)(p_1)_i, \\ F_{(p_2)_i}(p_1, p_2, z_1, z_2, x') &= \gamma_2(|p_2|)_{(p_2)_i} = b(|p_2|)(p_2)_i, \\ F_{(p_1)_i(p_1)_j}(p_1, p_2, z_1, z_2, x') &= a(|p_1|)\delta_{ij} + a'(|p_1|)\frac{(p_1)_i(p_1)_j}{|p_1|} = A_{ij}(p_1), \\ F_{(p_2)_i(p_2)_j}(p_1, p_2, z_1, z_2, x') &= b(|p_2|)\delta_{ij} + b'(|p_2|)\frac{(p_2)_i(p_2)_j}{|p_2|} = B_{ij}(p_2) \end{aligned}$$

$$\text{and } F_{(p_1)_i(p_2)_j}(p_1, p_2, z_1, z_2, x') = 0.$$

Therefore, Lemma 2.1 in [9] implies the desired convexity properties on F . Moreover, it also gives that $|F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')|$ and $|F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')|$ are bounded from above and below by $(\lambda_1(|p_1|) + \lambda_2(|p_1|))$ and $(\gamma_1(|p_2|) + \gamma_2(|p_2|))$, respectively, up to multiplicative constants.

Furthermore, since conditions (B1) and (B2) in [13] are satisfied for a , b , A and B , one can use Lemma 4.2 there to see that, if $|p_1|, |p_2| \leq M$, then

$$\begin{aligned} \lambda_1(|p_1|) &\leq C_M \lambda_2(|p_1|), & \lambda_2(|p_1 + q_1|) &\leq C_M \lambda_2(|p_1|), \\ \gamma_1(|p_2|) &\leq C_M \gamma_2(|p_2|) & \text{and} & \gamma_2(|p_2 + q_2|) \leq C_M \gamma_2(|p_2|), \end{aligned}$$

if $|q_1| \leq |p_1|/2$ and $|q_2| \leq |p_2|/2$, for some $C_M > 0$. As a consequence,

$$(7.6) \quad \begin{aligned} |F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')| &\leq C_M \lambda_2(|p_1|) \\ \text{and } |F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')| &\leq C_M \gamma_2(|p_2|), \end{aligned}$$

up to rename C_M . Therefore, recalling also formulas (1.4)-(1.7) in [9], we get

$$\begin{aligned} |F_{p_1 p_1}(p_1 + q_1, p_2, z_1, z_2, x')| &\leq C_M \lambda_2(|p_1 + q_1|) \\ &\leq C_M \lambda_2(|p_1|) \leq C_M |F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')| \\ \text{and } |F_{p_2 p_2}(p_1, p_2 + q_2, z_1, z_2, x')| &\leq C_M \gamma_2(|p_2 + q_2|) \\ &\leq C_M \gamma_2(|p_2|) \leq C_M |F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')|, \end{aligned}$$

for any $2|q_1| \leq |p_1| \leq M$ and $2|q_2| \leq |p_2| \leq M$. This means that (1.4) is satisfied.

It remains to check the growth condition (1.5). For this, notice that (7.6) and formula (4.13) in [13] give that

$$\begin{aligned} &|F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')| |p_1|^2 + |F_{p_2 p_2}(p_1, p_2, z_1, z_2, x')| |p_2|^2 \\ &\leq C_M (\lambda_2(|p_1|) |p_1|^2 + \gamma_2(|p_2|) |p_2|^2) \\ &= C_M (a(|p_1|) |p_1|^2 + b(|p_2|) |p_2|^2) \leq C_M (\Lambda_2(|p_1|) + \Gamma_2(|p_2|)). \end{aligned}$$

This and (7.5) imply that

$$\begin{aligned} &\int_{B_R} |F_{p_1 p_1}(\nabla u, \nabla v, u, v, x')| |\nabla u|^2 + |F_{p_2 p_2}(\nabla u, \nabla v, u, v, x')| |\nabla v|^2 dx \\ &\leq C_M \int_{B_R} (\Lambda_2(|\nabla u|) + \Gamma_2(|\nabla v|)) dx \leq C_M R^2, \end{aligned}$$

up to rename C_M . This shows (1.5) holds true.

Hence, we can apply Theorem 1.10, thus obtaining the desired monotonicity property. \square

As paradigmatic examples in Proposition 7.3 one can take the p -Laplacian, with $p \in (1, +\infty)$ if $\{\nabla u = 0\} = \emptyset$ and $p \in [2, +\infty)$ if $\{\nabla u = 0\} \neq \emptyset$ (in this case $a(t) = t^{p-2}$), or the mean curvature operator (in this case $a(t) = (1+t^2)^{-1/2}$). We stress on the fact that one can also take different operators a and b satisfying the hypotheses (for instance, one can take a to be the p -Laplacian and b an operator of mean curvature type).

Furthermore, we observe that condition (7.5) is satisfied, for instance, when $n = 2$ and ∇u and ∇v are bounded (thanks to the hypotheses on a and b , see page 3474 in [9]). This means that we recover Theorem 7.1 of [9] for stable solutions, without requiring conditions on the sign of \tilde{F}_{12} .

7.3. Systems involving the fractional Laplacian. The general setting of our results allows us to treat also the case of nonlocal systems of equations, i.e.

$$(7.7) \quad \begin{cases} (-\Delta)^{s_1} u = \tilde{F}_1(u, v), \\ (-\Delta)^{s_2} v = \tilde{F}_2(u, v), \end{cases}$$

where $s_1, s_2 \in (0, 1)$, \tilde{F} is a $C_{loc}^{1,1}$ function on \mathbb{R}^2 and \tilde{F}_1 and \tilde{F}_2 denote the derivatives of \tilde{F} with respect to the first and the second variable respectively.

As a matter of fact, as in Remark 2.12 of [18], we observe that one can generalize the functional considered in (1.1) to the following functional

$$(7.8) \quad \int_{\Omega} F(\nabla u, \nabla v, u, v, x') dx + \int_{\partial\Omega} G(u, v, x') d\mathcal{H}^{n-1},$$

where G satisfies the same regularity assumptions as F .

Furthermore, the growth condition in (1.5) can be weakened in the following way. We define $\ell_0(R) := R$ and

$$\ell_k(R) := \log(\ell_{k-1}(R)) = \underbrace{\log \circ \dots \circ \log R}_{k \text{ times}} \quad \text{for any } k \in \mathbb{N}, k \geq 1.$$

We also set

$$\pi_k(R) := \prod_{j=0}^k \ell_j(R).$$

Then, one can require that, for some $k \in \mathbb{N}$,

$$(7.9) \quad \int_{\Omega \cap B_R} |F_{p_1 p_1}(\nabla u, \nabla v, u, v, x')| |\nabla u|^2 + |F_{p_2 p_2}(\nabla u, \nabla v, u, v, x')| |\nabla v|^2 \\ + |F_{p_1 p_2}(\nabla u, \nabla v, u, v, x')| |\nabla u| |\nabla v| dx \leq CR \pi_k(R),$$

instead of (1.5). Notice that if $k = 0$ then (7.9) corresponds to (1.5). See Remark 2.13 and Section 9 in [18] for the proof and further discussion on this fact.

We also recall that, using the extension result in [7], one can localize problem (7.7) by adding one variable, see e.g. formula (1.7) in [10]. We will denote by U and V the extension functions of u and v , respectively (see e.g. formulas (2.3) and (2.4) in [10]).

So one has to deal with an energy functional like (7.8) with

$$(7.10) \quad \begin{aligned} \Omega &:= (0, +\infty) \times \mathbb{R}^n, \quad V := (0, +\infty) \times \mathbb{R}^{n-1}, \\ F(p_1, p_2, z_1, z_2, x') &:= x_1^{1-2s_1} |p_1|^2 + x_1^{1-2s_2} |p_2|^2 \\ \text{and } G(z_1, z_2, x') &:= \tilde{F}(z_1, z_2) \end{aligned}$$

(notice that in this application one has to replace n with $n + 1$ to apply Theorem 1.10).

With this, we can prove the following:

Proposition 7.4. *Let $u, v \in C^2(\mathbb{R}^n)$. Suppose that (u, v) is a stable solution to (7.7). Also, assume that the following growth condition holds true in the extension: there exist a constant $C > 0$ and $k \in \mathbb{N}$ such that, for R sufficiently large,*

$$(7.11) \quad \int_{B_R} x_1^{1-2s_1} |\nabla U|^2 + x_1^{1-2s_2} |\nabla V|^2 dx_1 \cdots dx_{n+1} \leq CR \pi_k(R).$$

Then (u, v) possesses one-dimensional symmetry.

Proof. With the notation introduced in (7.10), we observe that the thesis simply follows from (7.11) (that ensures the growth condition (7.9)) and Theorem 1.10. \square

We remark that (7.11) is a reasonable energy growth condition, since it is satisfied for instance in the case of a single equation when $n = 2$ for any $s \in (0, 1)$ and when $n = 3$ for $s \in (1/2, 1)$, see [3, 4]. Moreover, if $s_1 = s_2 = 1/2$, it can be checked as in [3] with suitable modifications under an additional assumption on the bound of ∇U and ∇V (see in particular formula (1.16) and Section 4 in [3]).

We finally remark that, differently from [10], we do not need here any sign assumption on \tilde{F} .

ACKNOWLEDGMENTS

The authors want to thank *Enrico Valdinoci* for very helpful discussions and comments.

This work has been supported by Alexander von Humboldt Foundation and ERC grant 277749 "EPSILON Elliptic PDE's and Symmetry of Interfaces and Layers for Odd Nonlinearities".

REFERENCES

- [1] H. BERESTYCKI, T.-C. LIN, J. WEI AND C. ZHAO: On phase-separation models: Asymptotics and qualitative properties. *Arch. Ration. Mech. Anal.* **208** (2013), no. 1, 163–200.
- [2] H. BERESTYCKI, S. TERRACINI, K. WANG, J. WEI: Existence and stability of entire solutions of an elliptic system modeling phase separation. *Adv. Math.* **243** (2013), 102–126.
- [3] X. CABRÉ, E. CINTI: Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian. *Discrete Contin. Dyn. Syst.* **28** (2010), no. 3, 1179–1206.
- [4] X. CABRÉ, E. CINTI: Sharp energy estimates for nonlinear fractional diffusion equations. *Calc. Var. Partial Differential Equations* **49** (2014), no. 1-2, 233–269.
- [5] L. CAFFARELLI, J.-M. ROQUEJOFFRE, O. SAVIN: Nonlocal minimal surfaces. *Comm. Pure Appl. Math.* **63** (2010), no. 9, 1111–1144.
- [6] L. CAFFARELLI, O. SAVIN, E. VALDINOCI: Minimization of a fractional perimeter-Dirichlet integral functional. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32** (2015), no. 4, 901–924.
- [7] L. CAFFARELLI, L. SILVESTRE: An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32** (2007), no. 7–9, 1245–1260.
- [8] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. math.* **136**, no. 5 (2012), 521–573.
- [9] S. DIPIERRO: Geometric inequalities and symmetry results for elliptic systems. *Discrete Contin. Dyn. Syst.* **33** (8) (2013), 3473–3496.
- [10] S. DIPIERRO, A. PINAMONTI: A geometric inequality and a symmetry result for elliptic systems involving the fractional Laplacian. *J. Differential Equations* **255** (2013), no. 1, 85–119.
- [11] S. DIPIERRO, O. SAVIN, E. VALDINOCI: A nonlocal free boundary problem. *SIAM J. Math. Anal.*, to appear, <http://arxiv.org/pdf/1411.7971.pdf>
- [12] A. FARINA: Some symmetry results for entire solutions of an elliptic system arising in phase separation. *Discrete Contin. Dyn. Syst.* **34** (6) (2014), 2505–2511.
- [13] A. FARINA, B. SCIUNZI, E. VALDINOCI: Bernstein and De Giorgi type problems: new results via a geometric approach. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **7**, no. 4, 741–791 (2008).
- [14] A. FARINA, N. SOAVE: Monotonicity and 1-dimensional symmetry for solutions of an elliptic system arising in Bose-Einstein condensation. *Arch. Ration. Mech. Anal.* **213** (2014), no. 1, 287–326.
- [15] M. FAZLY, N. GHOUSSOUB: De Giorgi type results for elliptic systems. *Calc. Var. Partial Differential Equations* **47** (2013), no. 3-4, 809–823.

- [16] M. FAZLY, Y. SIRE: Symmetry results for fractional elliptic systems and related problems. *Comm. Partial Differential Equations* **40** (2015), no. 6, 1070–1095.
- [17] B. NORIS, H. TAVARES, S. TERRACINI, G. VERZINI: Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. *Comm. Pure Appl. Math.* **63** (2010), no. 3, 267–302.
- [18] O. SAVIN, E. VALDINOCI: Some monotonicity results for minimizers in the calculus of variations. *J. Funct. Anal.* **264** (2013), no. 10, 2469–2496.
- [19] O. SAVIN, E. VALDINOCI: Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations* **48** (2013), no. 1-2, 33–39.
- [20] S. TERRACINI, G. VERZINI, A. ZILIO: Uniform Hölder bounds for strongly competing systems involving the square root of the Laplacian. *Preprint* (2012), <http://arxiv.org/pdf/1211.6087.pdf>
- [21] S. TERRACINI, G. VERZINI, A. ZILIO: Uniform Hölder regularity with small exponent in competition-fractional diffusion systems. *Discrete Contin. Dyn. Syst.* **34** (2014), no. 6, 2669–2691.
- [22] N. TRUDINGER: Hölder gradient estimates for fully nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh, Sect. A* **108** (1988), 57–65.
- [23] G. VERZINI, A. ZILIO: Strong competition versus fractional diffusion: the case of Lotka-Volterra interaction. *Comm. Partial Differential Equations* **39** (2014), no. 1-2, 2284–2313.
- [24] K. WANG: On the De Giorgi type conjecture for an elliptic system modeling phase separation. *Comm. Partial Differential Equations* **39** (4) (2014), 696–739.
- [25] K. WANG: Harmonic approximation and improvement of flatness in a singular perturbation problem. *Manuscripta Math.* **146** (1-2) (2015), 281–298.

(Julien Brasseur) INSTITUT DE MATHÉMATIQUES DE MARSEILLE (I2M), CENTRE DE MATHÉMATIQUES ET INFORMATIQUE (CMI), TECHNOPÔLE CHÂTEAU-GOMBERT, 39, RUE F. JOLIOT CURIE, 13453 MARSEILLE CEDEX 13, FRANCE

E-mail address: `julien.brasseur@ens-lyon.fr`

(Serena Dipierro) INSTITUT FÜR ANALYSIS UND NUMERIK, OTTO-VON-GUERICKE UNIVERSITÄT MAGDEBURG, UNIVERSITÄTSPLATZ 2, 39106 MAGDEBURG, GERMANY

E-mail address: `serydipierro@yahoo.it`